

Frobenius restricted varieties in numerical semigroups

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Abstract

The common behaviour of many families of numerical semigroups led up to defining, firstly, the Frobenius varieties and, secondly, the (Frobenius) pseudo-varieties. However, some interesting families are still out of these definitions. To overcome this situation, here we introduce the concept of Frobenius restricted variety (or R -variety). We will generalize most of the results for varieties and pseudo-varieties to R -varieties. In particular, we will study the tree structure that arise within them.

Keywords: R -varieties; Frobenius restricted number; varieties; pseudo-varieties; monoids; numerical semigroups; tree (associated to an R -variety).

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1 Introduction

In [11], the concept of (Frobenius) variety was introduced in order to unify several results which have appeared in [1], [3], [16], and [17]. Moreover, the work made in [11] has allowed to study other notables families of numerical semigroups, such as those that appear in [7], [9], [12], and [13].

There exist families of numerical semigroups which are not varieties but have a similar structure. For example, the family of numerical semigroups with maximal embedding dimension and fixed multiplicity (see [15]). The study of this family, in [2], led to the concept of m -variety.

In order to generalize the concepts of variety and m -variety, in [8] were introduced the (Frobenius) pseudo-varieties. Moreover, recently, the results obtained in [8] allowed us to study several interesting families of numerical semigroups (for instance, see [10]).

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In this work, our aim will be to introduce and study the concept of R -variety (that is, Frobenius restricted variety). We will see how it generalizes the concept of pseudo-variety and we will show that there exist significant families of numerical semigroups which are R -varieties but not pseudo-varieties.

Let \mathbb{N} be the set of nonnegative integers. A *numerical semigroup* is a subset S of \mathbb{N} such that it is closed under addition, contains the zero element, and $\mathbb{N} \setminus S$ is finite.

It is well known (see [14, Lemma 4.5]) that, if S and T are numerical semigroups such that $S \subsetneq T$, then $S \cup \{\max(T \setminus S)\}$ is another numerical semigroup. We will denote by $F_T(S) = \max(T \setminus S)$ and we will call it as the *Frobenius number of S restricted to T* .

An R -variety is a non-empty family \mathcal{R} of numerical semigroups that fulfills the following conditions.

1. \mathcal{R} has a maximum element with respect to the inclusion order (that we will denote by $\Delta(\mathcal{R})$).
2. If $S, T \in \mathcal{R}$, then $S \cap T \in \mathcal{R}$.
3. If $S \in \mathcal{R}$ and $S \neq \Delta(\mathcal{R})$, then $S \cup \{F_{\Delta(\mathcal{R})}(S)\} \in \mathcal{R}$.

In Section 2 we will see that every pseudo-variety is an R -variety. Moreover, we will show that, if \mathcal{V} is a variety and T is a numerical semigroup, then $\mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\}$ is an R -variety. In fact, we will prove that every R -variety is of this form.

Let \mathcal{R} be an R -variety and let M be a submonoid of $(\mathbb{N}, +)$. We will say that M is an \mathcal{R} -monoid if it can be expressed as intersection of elements of \mathcal{R} . It is clear that the intersection of \mathcal{R} -monoids is another \mathcal{R} -monoid and, therefore, we can define the \mathcal{R} -monoid generated by a subset of $\Delta(\mathcal{R})$. In Section 3 we will show that every \mathcal{R} -monoid admits a unique minimal \mathcal{R} -system of generators. In addition, we will see that, if M is an \mathcal{R} -monoid and $x \in M$, then $M \setminus \{x\}$ is another \mathcal{R} -monoid if and only if x belongs to the minimal \mathcal{R} -system of generators of M .

In Section 4 we will show that the elements of an R -variety, \mathcal{R} , can be arranged in a tree with root $\Delta(\mathcal{R})$. Moreover, we will prove that the set of children of a vertex S , of such a tree, is equal to $\{S \setminus \{x\} \mid x \text{ is an element of the minimal } \mathcal{R}\text{-system of generators of } S \text{ and } x > F_{\Delta(\mathcal{R})}(S)\}$. This fact will allow us to show an algorithmic process in order to recurrently build the elements of an R -variety.

Finally, in Section 5 we will see that, in general and contrary to what happens with varieties and pseudo-varieties, we cannot define the smallest R -variety that contains a given family \mathcal{F} of numerical semigroups. Nevertheless, we will show that, if Δ is a numerical semigroup such that $S \subseteq \Delta$ for all $S \in \mathcal{F}$, then there exists the smallest R -variety (denoted by $\mathcal{R}(\mathcal{F}, \Delta)$) containing \mathcal{F} and having Δ as maximum (with respect the inclusion order). Moreover, we will prove that $\mathcal{R}(\mathcal{F}, \Delta)$ is finite if and only if \mathcal{F} is finite. In such a case, that fact will allow us to compute, for a given $\mathcal{R}(\mathcal{F}, \Delta)$ -monoid, its minimal $\mathcal{R}(\mathcal{F}, \Delta)$ -system of

generators. In this way, we will obtain an algorithmic process to determine all the elements of $\mathcal{R}(\mathcal{F}, \Delta)$ by starting from \mathcal{F} and Δ .

Let us observe that the proofs, of some results of this work, are similar to the proofs of the analogous results for varieties and pseudo-varieties. However, in order to get a self-contained paper, we have not omitted several of such proofs.

2 Varieties, pseudo-varieties, and R -varieties

It is said that M is a *submonoid* of $(\mathbb{N}, +)$ if M is a subset of \mathbb{N} which is closed for the addition and such that $0 \in M$. In particular, if S is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N} \setminus S$ is finite, then S is a numerical semigroup.

Let A be a non-empty subset of \mathbb{N} . Then it is denoted by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A , that is,

$$\langle A \rangle = \{ \lambda_1 a_1 + \dots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, a_1, \dots, a_n \in A, \lambda_1, \dots, \lambda_n \in \mathbb{N} \}.$$

It is well known (see for instance [14, Lemma 2.1]) that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

Let M be a submonoid of $(\mathbb{N}, +)$ and let $A \subseteq \mathbb{N}$. If $M = \langle A \rangle$, then it is said that A is a *system of generators* of M . Moreover, it is said that A is a *minimal system of generators* of M if $M \neq \langle B \rangle$ for all $B \subsetneq A$. It is a classical result that every submonoid M of $(\mathbb{N}, +)$ has a unique minimal system of generators (denoted by $\text{msg}(M)$) which, in addition, is finite (see for instance [14, Corollary 2.8]).

Let S be a numerical semigroup. Being that $\mathbb{N} \setminus S$ is finite, it is possible to define several notable invariants of S . One of them is the *Frobenius number* of S (denoted by $F(S)$) which is the greatest integer that does not belong to S (see [6]). Another one is the *genus* of S (denoted by $g(S)$) which is the cardinality of $\mathbb{N} \setminus S$.

Let S be a numerical semigroup different from \mathbb{N} . Then it is obvious that $S \cup \{F(S)\}$ is also a numerical semigroup. Moreover, from [14, Proposition 7.1], we have that T is a numerical semigroup with $g(T) = g + 1$ if and only if there exist a numerical semigroup S and $x \in \text{msg}(S)$ such that $g(S) = g$, $x > F(S)$, and $T = S \setminus \{x\}$. This result is the key to build the set of all numerical semigroups with genus $g + 1$ when we have the set of all numerical semigroups with genus g (see [14, Proposition 7.4]).

In [11] it was introduced the concept of *(Frobenius) variety* in order to generalize the previous situation to some relevant families of numerical semigroups.

It is said that a non-empty family of numerical semigroups \mathcal{V} is a *(Frobenius) variety* if the following conditions are verified.

1. If $S, T \in \mathcal{V}$, then $S \cap T \in \mathcal{V}$.
2. If $S \in \mathcal{V}$ and $S \neq \mathbb{N}$, then $S \cup \{F(S)\} \in \mathcal{V}$.

However, there exist families of numerical semigroups that are not varieties, but have a very similar behavior. By studying these families of numerical semigroups, we introduced in [8] the concept of *(Frobenius) pseudo-variety*.

It is said that a non-empty family of numerical semigroups \mathcal{P} is a (*Frobenius*) *pseudo-variety* if the following conditions are verified.

1. \mathcal{P} has a maximum element with respect to the inclusion order (that we will denote by $\Delta(\mathcal{P})$).
2. If $S, T \in \mathcal{P}$, then $S \cap T \in \mathcal{P}$.
3. If $S \in \mathcal{P}$ and $S \neq \Delta(\mathcal{P})$, then $S \cup \{F(S)\} \in \mathcal{P}$.

From the definitions, it is clear that every variety is a pseudo-variety. Moreover, as a consequence of [8, Proposition 1], we have the next result.

Proposition 2.1. *Let \mathcal{P} be a pseudo-variety. Then \mathcal{P} is a variety if and only if $\mathbb{N} \in \mathcal{P}$.*

The following result asserts that the concept of R -variety generalizes the concept of pseudo-variety.

Proposition 2.2. *Every pseudo-variety is an R -variety.*

Proof. Let \mathcal{P} be a pseudo-variety. In order to prove that \mathcal{P} is an R -variety, we have to show that, if $S \in \mathcal{P}$ and $S \neq \Delta(\mathcal{P})$, then $S \cup \{F_{\Delta(\mathcal{P})}(S)\} \in \mathcal{P}$. Since \mathcal{P} is a pseudo-variety, we know that $S \cup \{F(S)\} \in \mathcal{P}$. Thus, to finish the proof, it is enough to see that $F(S) = F_{\Delta(\mathcal{P})}(S)$. On the one hand, it is clear that $F_{\Delta(\mathcal{P})}(S) \leq F(S)$. On the other hand, since $S \cup \{F(S)\} \in \mathcal{P}$, then we have that $F(S) \in \Delta(\mathcal{P})$. Therefore, $F(S) \in \Delta(\mathcal{P}) \setminus S$ and, consequently, $F(S) \leq F_{\Delta(\mathcal{P})}(S)$. \square

In the next example we see that there exist R -varieties that are not pseudo-varieties.

Example 2.3. Let \mathcal{R} be the set formed by all numerical semigroups which are contained in the numerical semigroup $\langle 5, 7, 9 \rangle$. It is clear that \mathcal{R} is an R -variety. However, since $S = \langle 5, 7, 9 \rangle \setminus \{5\} \in \mathcal{R}$, $S \neq \Delta(\mathcal{R}) = \langle 5, 7, 9 \rangle$, $F(S) = 13$, and $S \cup \{13\} \notin \mathcal{R}$, we have that \mathcal{R} is not a pseudo-variety.

Generalizing the above example, we can obtain several R -varieties, most of which are not pseudo-varieties.

1. Let T be a numerical semigroup. Then $\mathcal{L}_T = \{S \mid S \text{ is a numerical semigroup and } S \subseteq T\}$ is an R -variety. Observe that \mathcal{L}_T is the set formed by all numerical subsemigroups of T .
2. Let S_1 and S_2 be two numerical semigroups such that $S_1 \subseteq S_2$. Then $[S_1, S_2] = \{S \mid S \text{ is a numerical semigroup and } S_1 \subseteq S \subseteq S_2\}$ is an R -variety.
3. Let T be a numerical semigroup and let $A \subseteq T$. Then $\mathcal{R}(A, T) = \{S \mid S \text{ is a numerical semigroup and } A \subseteq S \subseteq T\}$ is an R -variety. Observe that both of the previous examples are particular cases of this one.

Remark 2.4. Let p, q be relatively prime integers such that $1 < p < q$. Let us take the numerical semigroups $S_1 = \langle p, q \rangle$ and $S_2 = \frac{S_1}{2} = \{s \in \mathbb{N} \mid 2s \in S_1\}$. In [4, 5], Kunz and Waldi study the family of numerical semigroups $[S_1, S_2]$, which is an R -variety but not a pseudo-variety.

The next result establishes when an R -variety is a pseudo-variety.

Proposition 2.5. *Let \mathcal{R} be an R -variety. Then \mathcal{R} is a pseudo-variety if and only if $F(S) \in \Delta(\mathcal{R})$ for all $S \in \mathcal{R}$ such that $S \neq \Delta(\mathcal{R})$.*

Proof. (Necessity.) If \mathcal{R} is a pseudo-variety and $S \in \mathcal{R}$ with $S \neq \Delta(\mathcal{R})$, then $S \cup \{F(S)\} \in \mathcal{R}$. Therefore, $F(S) \in \Delta(\mathcal{R})$.

(Sufficiency.) In order to show that \mathcal{R} is a pseudo-variety, it will be enough to see that $S \cup \{F(S)\} \in \mathcal{R}$ for all $S \in \mathcal{R}$ such that $S \neq \Delta(\mathcal{R})$. For that, since $F(S) \in \Delta(\mathcal{R})$, then it is clear that $F_{\Delta(\mathcal{R})}(S) = F(S)$ and, therefore, $S \cup \{F(S)\} = S \cup \{F_{\Delta(\mathcal{R})}(S)\} \in \mathcal{R}$. \square

An immediate consequence of Propositions 2.1 and 2.5 is the following result.

Corollary 2.6. *Let \mathcal{R} be an R -variety. Then \mathcal{R} is a variety if and only if $\mathbb{N} \in \mathcal{R}$.*

Our next purpose, in this section, will be to show that to give an R -variety is equivalent to give a pair (\mathcal{V}, T) where \mathcal{V} is a variety and T is a numerical semigroup. Before that we need to introduce some concepts and results.

Let S be a numerical semigroup. Then we define recurrently the following sequence of numerical semigroups.

- $S_0 = S$,
- if $S_i \neq \mathbb{N}$, then $S_{i+1} = S_i \cup \{F(S_i)\}$.

Since $\mathbb{N} \setminus S$ is a finite set with cardinality equal to $g(S)$, then we get a finite chain of numerical semigroups $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_{g(S)} = \mathbb{N}$. We will denote by $C(S)$ the set $\{S_0, S_1, \dots, S_{g(S)}\}$ and will say that it is the *chain of numerical semigroups associated to S* . If \mathcal{F} is a non-empty family of numerical semigroups, then we will denote by $C(\mathcal{F})$ the set $\bigcup_{S \in \mathcal{F}} C(S)$.

Let \mathcal{F} be a non-empty family of numerical semigroups. We know that there exists the smallest variety containing \mathcal{F} (see [11]). Moreover, by [11, Theorem 4], we have the next result.

Proposition 2.7. *Let \mathcal{F} be a non-empty family of numerical semigroups. Then the smallest variety containing \mathcal{F} is the set formed by all finite intersections of elements of $C(\mathcal{F})$.*

Now, let \mathcal{R} be an R -variety. By applying repeatedly that, if $S \in \mathcal{R}$ and $S \neq \Delta(\mathcal{R})$, then $S \cup \{F_{\Delta(\mathcal{R})}(S)\} \in \mathcal{R}$, we get the following result.

Lemma 2.8. *Let \mathcal{R} be an R -variety. If $S \in \mathcal{R}$ and $n \in \mathbb{N}$, then $S \cup \{x \in \Delta(\mathcal{R}) \mid x \geq n\} \in \mathcal{R}$.*

We are ready to show the announced result.

Theorem 2.9. *Let \mathcal{V} be a variety and let T be a numerical semigroup. Then $\mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\}$ is an R -variety. Moreover, every R -variety is of this form.*

Proof. By Proposition 2.1, we know that, if \mathcal{V} is a variety, then $\mathbb{N} \in \mathcal{V}$ and, therefore, T is the maximum of \mathcal{V}_T (that is, $T = \Delta(\mathcal{V}_T)$). On the other hand, it is clear that, if $S_1, S_2 \in \mathcal{V}_T$, then $S_1 \cap S_2 \in \mathcal{V}_T$.

Now, let $S \in \mathcal{V}$ such that $S \cap T \neq T$ and let us have $t = F_T(S \cap T)$. In order to conclude that \mathcal{V}_T is an R -variety, we will see that $(S \cap T) \cup \{t\} \in \mathcal{V}_T$. First, let us observe that $t = \max(T \setminus (S \cap T)) = \max(T \setminus S)$. Then, because $S \in \mathcal{V}$ and \mathcal{V} is a variety, we can easily deduce that $\bar{S} = S \cup \{t, \rightarrow\} \in \mathcal{V}$. Moreover, $(S \cap T) \cup \{t\} \subseteq (S \cap T) \cup (\{t, \rightarrow\} \cap T) = \bar{S} \cap T$. Let us see now that $\bar{S} \cap T \subseteq (S \cap T) \cup \{t\}$. In other case, there exists $t' > t$ such that $t' \in T$ and $t' \notin S$, in contradiction with the maximality of t . Therefore, $(S \cap T) \cup \{t\} = \bar{S} \cap T$ and $\bar{S} \in \mathcal{V}$. Consequently, $(S \cap T) \cup \{t\} \in \mathcal{V}_T$.

Let \mathcal{R} be an R -variety and let \mathcal{V} be the smallest variety containing \mathcal{R} . To conclude the proof of the theorem, we will see that $\mathcal{R} = \mathcal{V}_{\Delta(\mathcal{R})}$. It is clear that $\mathcal{R} \subseteq \mathcal{V}_{\Delta(\mathcal{R})}$. Thus, let us see the reverse one. For that, we will prove that, if $S \in \mathcal{V}$, then $S \cap \Delta(\mathcal{R}) \in \mathcal{R}$. In effect, by Proposition 2.7 we have that, if $S \in \mathcal{V}$, then there exist $S_1, \dots, S_k \in \mathcal{C}(\mathcal{R})$ such that $S = S_1 \cap \dots \cap S_k$. Therefore, $S \cap \Delta(\mathcal{R}) = (S_1 \cap \Delta(\mathcal{R})) \cap \dots \cap (S_k \cap \Delta(\mathcal{R}))$. Since \mathcal{R} is an R -variety, then \mathcal{R} is closed under finite intersections. Thereby, to see that $S \cap \Delta(\mathcal{R}) \in \mathcal{R}$, it is enough to show that $S_i \cap \Delta(\mathcal{R}) \in \mathcal{R}$ for all $i \in \{1, \dots, k\}$. Since $S_i \in \mathcal{C}(\mathcal{R})$, then it is clear that there exist $S'_i \in \mathcal{R}$ and $n_i \in \mathbb{N}$ such that $S_i = S'_i \cup \{n_i, \rightarrow\}$. Therefore, $S_i \cap \Delta(\mathcal{R}) = S'_i \cup \{x \in \Delta(\mathcal{R}) \mid x \geq n_i\} \in \mathcal{R}$, by applying Lemma 2.8. \square

The above theorem allows us to give many examples of R -varieties starting from already known varieties.

1. Let us observe that, if \mathcal{V} is a variety and $T \in \mathcal{V}$, then $\mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\} = \{S \in \mathcal{V} \mid S \subseteq T\}$ is an R -variety contained in \mathcal{V} . Thus, for instance, we have that the set formed by all Arf numerical semigroups, which are contained in a certain Arf numerical semigroup, is an R -variety.
2. Observe also that, if \mathcal{V} is a variety and T is a numerical semigroup such that $T \notin \mathcal{V}$, then $\mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\}$ is an R -variety not contained in \mathcal{V} (because $T \in \mathcal{V}_T$ and $T \notin \mathcal{V}$). Let us take, for example, the variety \mathcal{V} of all Arf numerical semigroups and $T = \langle 5, 8 \rangle \notin \mathcal{V}$. In such a case, \mathcal{V}_T is the R -variety formed by the numerical semigroups which are the intersection of an Arf numerical semigroup and T .

Corollary 2.10. *Let \mathcal{R} be an R -variety and let U be a numerical semigroup. Then $\mathcal{R}_U = \{S \cap U \mid S \in \mathcal{R}\}$ is an R -variety.*

Proof. By applying Theorem 2.9, we have that there exist a variety \mathcal{V} and a numerical semigroup T such that $\mathcal{R} = \mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\}$. Therefore,

$\mathcal{R}_U = \{S \cap T \cap U \mid S \in \mathcal{V}\} = \mathcal{V}_{T \cap U}$, which is clearly an R -variety (by Theorem 2.9 again). \square

The next result says us that Theorem 2.9 remains true when variety is replaced with pseudo-variety.

Corollary 2.11. *Let \mathcal{P} be a pseudo-variety and let T be a numerical semigroup. Then $\mathcal{P}_T = \{S \cap T \mid S \in \mathcal{P}\}$ is an R -variety. Moreover, every R -variety is of this form.*

Proof. By Proposition 2.2, we know that, if \mathcal{P} is a pseudo-variety, then \mathcal{P} is an R -variety. Thereby, by applying Corollary 2.10, we conclude that \mathcal{P}_T is an R -variety.

Now, by Theorem 2.9, we know that, if \mathcal{R} is an R -variety, then there exist a variety \mathcal{V} and a numerical semigroup T such that $\mathcal{R} = \mathcal{V}_T$. To finish the proof, it is enough to observe that all varieties are pseudo-varieties. \square

Let us see an illustrative example of the above corollary.

Example 2.12. From [8, Example 7], we have the pseudo-variety

$$\mathcal{P} = \{\langle 5, 6, 8, 9 \rangle, \langle 5, 6, 9, 13 \rangle, \langle 5, 6, 8 \rangle, \langle 5, 6, 13, 14 \rangle, \\ \langle 5, 6, 9 \rangle, \langle 5, 6, 14 \rangle, \langle 5, 6, 13 \rangle, \langle 5, 6, 19 \rangle, \langle 5, 6 \rangle\}.$$

Thereby, we have that \mathcal{P}_T is an R -variety for each numerical semigroup T .

3 Monoids associated to an R -variety

In this section, \mathcal{R} will be an R -variety. Now, let M be a submonoid of $(\mathbb{N}, +)$. We will say that M is an \mathcal{R} -monoid if it is the intersection of elements of \mathcal{R} . The next result is easy to proof.

Lemma 3.1. *The intersection of \mathcal{R} -monoids is an \mathcal{R} -monoid.*

From the above lemma we have the following definition: let $A \subseteq \Delta(\mathcal{R})$. We will say that $\mathcal{R}(A)$ is the \mathcal{R} -monoid generated by A if $\mathcal{R}(A)$ is equal to the intersection of all the \mathcal{R} -monoids which contain the set A . Observe that $\mathcal{R}(A)$ is the smallest \mathcal{R} -monoid which contains the set A (with respect to the inclusion order). The next result has an easy proof too.

Lemma 3.2. *If $A \subseteq \Delta(\mathcal{R})$, then $\mathcal{R}(A)$ is equal to the intersection of all the elements of \mathcal{R} which contain the set A .*

Let us take $A \subseteq \Delta(\mathcal{R})$. If $M = \mathcal{R}(A)$, then we will say that A is an \mathcal{R} -system of generators of M . Moreover, we will say that A is a *minimal \mathcal{R} -system of generators* of M if $M \neq \mathcal{R}(B)$ for all $B \subsetneq A$. The next purpose in this section will be to show that every \mathcal{R} -monoid has a unique minimal \mathcal{R} -system of generators. For that, we will give some previous lemmas. We can easily deduced the first one from Lemma 3.2.

Lemma 3.3. *Let A, B be two subsets of $\Delta(\mathcal{R})$ and let M be an \mathcal{R} -monoid. We have that*

1. *if $A \subseteq B$, then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;*
2. *$\mathcal{R}(A) = \mathcal{R}(\langle A \rangle)$;*
3. *$\mathcal{R}(M) = M$.*

If M is an \mathcal{R} -monoid, then M is a submonoid of $(\mathbb{N}, +)$. Moreover, as we commented in Section 2, we know that there exists a finite subset A of M such that $M = \langle A \rangle$. Thereby, by applying Lemma 3.3, we have that $M = \mathcal{R}(M) = \mathcal{R}(\langle A \rangle) = \mathcal{R}(A)$. Consequently, A is a finite \mathcal{R} -system of generators of M . Thus, we can establish the next result.

Lemma 3.4. *Every \mathcal{R} -monoid has a finite \mathcal{R} -system of generators.*

In the following result, we characterize the minimal \mathcal{R} -systems of generators.

Lemma 3.5. *Let $A \subseteq \Delta(\mathcal{R})$ and $M = \mathcal{R}(A)$. Then A is a minimal \mathcal{R} -system of generators of M if and only if $a \notin \mathcal{R}(A \setminus \{a\})$ for all $a \in A$.*

Proof. (Necessity.) If $a \in \mathcal{R}(A \setminus \{a\})$, then $A \subseteq \mathcal{R}(A \setminus \{a\})$. Thus, by Lemma 3.3, we get that $M = \mathcal{R}(A) \subseteq \mathcal{R}(\mathcal{R}(A \setminus \{a\})) = \mathcal{R}(A \setminus \{a\}) \subseteq \mathcal{R}(A) = M$. Therefore, $M = \mathcal{R}(A \setminus \{a\})$, in contradiction with the minimality of A .

(Sufficiency.) If A is not a minimal \mathcal{R} -system of generators of M , then there exists $B \subsetneq A$ such that $\mathcal{R}(B) = M$. Then, by Lemma 3.3, if $a \in A \setminus B$, then $a \in M = \mathcal{R}(B) \subseteq \mathcal{R}(A \setminus \{a\})$, in contradiction with the hypothesis. \square

The next result generalizes an evident property of submonoids of $(\mathbb{N}, +)$. More concretely, every element x of a submonoid M of $(\mathbb{N}, +)$ is expressible as a non-negative integer linear combination of the generators of M that are smaller than or equal to x .

Lemma 3.6. *Let $A \subseteq \Delta(\mathcal{R})$ and $x \in \mathcal{R}(A)$. Then $x \in \mathcal{R}(\{a \in A \mid a \leq x\})$.*

Proof. Let us suppose that $x \notin \mathcal{R}(\{a \in A \mid a \leq x\})$. Then, from Lemma 3.2, we know that there exists $S \in \mathcal{R}$ such that $\{a \in A \mid a \leq x\} \subseteq S$ and $x \notin S$. By applying now Lemma 2.8, we have that $\bar{S} = S \cup \{m \in \Delta(\mathcal{R}) \mid m \geq x + 1\} \in \mathcal{R}$. Observe that, obviously, $A \subseteq \bar{S}$ and $x \notin \bar{S}$. Therefore, by applying once again Lemma 3.2, we get that $x \notin \mathcal{R}(A)$, in contradiction with the hypothesis. \square

We are now ready to show the above announced result.

Theorem 3.7. *Every \mathcal{R} -monoid admits a unique minimal \mathcal{R} -system of generators. In addition, such a \mathcal{R} -system is finite.*

Proof. Let M be an \mathcal{R} -monoid and let A, B be two minimal \mathcal{R} -systems of generators of M . We are going to see that $A = B$. For that, let us suppose that $A = \{a_1 < a_2 < \dots\}$ and $B = \{b_1 < b_2 < \dots\}$. If $A \neq B$, then

there exists $i = \min\{k \mid a_k \neq b_k\}$. Let us assume, without loss of generality, that $a_i < b_i$. Since $a_i \in M = \mathcal{R}(A) = \mathcal{R}(B)$, by Lemma 3.6, we have that $a_i \in \mathcal{R}(\{b_1, \dots, b_{i-1}\})$. Because $\{b_1, \dots, b_{i-1}\} = \{a_1, \dots, a_{i-1}\}$, then $a_i \in \mathcal{R}(\{a_1, \dots, a_{i-1}\})$, in contradiction with Lemma 3.5. Finally, by Lemma 3.4, we have that the minimal \mathcal{R} -system of generators is finite. \square

If M is a \mathcal{R} -monoid, then the cardinality of the minimal \mathcal{R} -system of generators of M will be called the \mathcal{R} -range of M .

Example 3.8. Let S, T be two numerical semigroups such that $S \subseteq T$. We define recurrently the following sequence of numerical semigroups.

- $S_0 = S$,
- if $S_i \neq T$, then $S_{i+1} = S_i \cup \{F_T(S_i)\}$.

Since $T \setminus S$ is a finite set, then we get a finite chain of numerical semigroups $S = S_0 \subsetneq S_1 \subsetneq \dots \subsetneq S_n = T$. We will denote by $C(S, T)$ the set $\{S_0, S_1, \dots, S_n\}$ and will say that it is the *chain of S restricted to T* . It is clear that $C(S, T)$ is an R -variety. Moreover, it is also clear that, for each $i \in \{1, \dots, n\}$, S_i is the smallest element of $C(S, T)$ containing $F_T(S_{i-1})$. Therefore, $\{F_T(S_{i-1})\}$ is the minimal $C(S, T)$ -system of generators of S_i for all $i \in \{1, \dots, n\}$. Let us also observe that the empty set, \emptyset , is the minimal $C(S, T)$ -system of generators of S_0 . Thereby, the $C(S, T)$ -range of S_i is equal to 1, if $i \in \{1, \dots, n\}$, and 0, if $i = 0$.

It is well known that, if M is a submonoid of $(\mathbb{N}, +)$ and $x \in M$, then $M \setminus \{x\}$ is another submonoid of $(\mathbb{N}, +)$ if and only if $x \in \text{msg}(M)$. In the next result we generalize this property to \mathcal{R} -monoids.

Proposition 3.9. *Let M be an \mathcal{R} -monoid and let $x \in M$. Then $M \setminus \{x\}$ is an \mathcal{R} -monoid if and only if x belongs to the minimal \mathcal{R} -system of generators of M .*

Proof. Let A be the minimal \mathcal{R} -system of generators of M . If $x \notin A$, then $A \subseteq M \setminus \{x\}$. Therefore, $M \setminus \{x\}$ is a \mathcal{R} -monoid containing A and, consequently, $M = \mathcal{R}(A) \subseteq M \setminus \{x\}$, which is a contradiction.

Conversely, by Theorem 3.7, we have that, if $x \in A$, then $\mathcal{R}(M \setminus \{x\}) \neq \mathcal{R}(A) = M$. Thereby, $\mathcal{R}(M \setminus \{x\}) = M \setminus \{x\}$. Consequently, $M \setminus \{x\}$ is a \mathcal{R} -monoid. \square

Let us illustrate the above proposition with an example.

Example 3.10. Let T be a numerical semigroup and let $A \subseteq T$. Then we know that $\mathcal{R}(A, T) = \{S \mid S \text{ is a numerical semigroup and } A \subseteq S \subseteq T\}$ is an R -variety. By applying Proposition 3.9, we easily deduce that, if $S \in \mathcal{R}(A, T)$, then the minimal $\mathcal{R}(A, T)$ -system of generators of S is $\{x \in \text{msg} \mid x \notin A\}$.

From Theorem 2.9 we know that every R -variety is of the form $\mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\}$, where \mathcal{V} is a variety and T is a numerical semigroup. Now, our purpose is to study the relation between \mathcal{V} -monoids and \mathcal{V}_T -monoids.

Proposition 3.11. *Let M be a submonoid of $(\mathbb{N}, +)$ and let T be a numerical semigroup. Then M is a \mathcal{V}_T -monoid if and only if there exists a \mathcal{V} -monoid M' such that $M = M' \cap T$.*

Proof. (Necessity.) If M is a \mathcal{V}_T -monoid, then there exists $\mathcal{F} \subseteq \mathcal{V}_T$ such that $M = \bigcap_{S \in \mathcal{F}} S$. But, if $S \in \mathcal{F}$, then $S \in \mathcal{V}_T$ and, consequently, there exists $S' \in \mathcal{V}$ such that $S = S' \cap T$. Now, let $\mathcal{F}' = \{S' \in \mathcal{V} \mid S' \cap T \in \mathcal{F}\}$ and let $M' = \bigcap_{S' \in \mathcal{F}'} S'$. Then it is clear that M' is a \mathcal{V} -monoid and that $M = M' \cap T$.

(Sufficiency.) If M' is a \mathcal{V} -monoid, then there exists $\mathcal{F}' \in \mathcal{V}$ such that $M' = \bigcap_{S' \in \mathcal{F}'} S'$. Let $\mathcal{F} = \{S' \cap T \mid S' \in \mathcal{F}'\}$. Then it is clear that $\mathcal{F} \subseteq \mathcal{V}_T$ and that $\bigcap_{S \in \mathcal{F}} S = M' \cap T$. Therefore, $M' \cap T$ is a \mathcal{V}_T -monoid. \square

Observe that, as a consequence of the above proposition, we have that the set of \mathcal{V}_T -monoids is precisely given by $\{M \cap T \mid M \text{ is a } \mathcal{V}\text{-monoid}\}$.

Corollary 3.12. *Let T be a numerical semigroup. If $A \subseteq T$, then $\mathcal{V}_T(A) = \mathcal{V}(A) \cap T$.*

Proof. By Proposition 3.11, we know that $\mathcal{V}(A) \cap T$ is a \mathcal{V}_T -monoid containing A . Therefore, $\mathcal{V}_T(A) \subseteq \mathcal{V}(A) \cap T$.

Let us see now the opposite inclusion. By applying once more Proposition 3.11, we deduce that there exists a \mathcal{V} -monoid M such that $\mathcal{V}_T(A) = M \cap T$. Thus, it is clear that $A \subseteq M$ and, thereby, $\mathcal{V}(A) \subseteq M$. Consequently, $\mathcal{V}(A) \cap T \subseteq M \cap T = \mathcal{V}_T(A)$. \square

From Corollary 3.12, we have that the set formed by the \mathcal{V}_T -monoids is $\{\mathcal{V}(A) \cap T \mid A \subseteq T\} = \{M \cap T \mid M \text{ is a } \mathcal{V}\text{-monoid and its minimal } \mathcal{V}\text{-system of generators is including in } T\}$. Moreover, observe that, if $T \in \mathcal{V}$, then $\mathcal{V}_T(A) = \mathcal{V}(A)$ and, therefore, in such a case the set formed by all the \mathcal{V}_T -monoids coincides with the set formed by all the \mathcal{V} -monoids that are contained in T .

For some varieties there exist algorithms that allow us to compute $\mathcal{V}(A)$ by starting from A . Thereby, we can use such results in order to compute $\mathcal{V}_T(A)$. Let us see two examples of this fact.

Example 3.13. An LD-semigroup (see [12]) is a numerical semigroup S fulfilling that $a + b - 1 \in S$ for all $a, b \in S \setminus \{0\}$. Let \mathcal{V} the set formed by all LD-semigroups. In [12] it is shown that \mathcal{V} is a variety. Let $T = \langle 5, 7, 9 \rangle$ (observe that $T \notin \mathcal{V}$). By Theorem 2.9, we know that $\mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\}$ is an R -variety. Let us suppose that we can compute $\mathcal{V}_T(\{5\})$.

In [12] we have an algorithm to compute $\mathcal{V}(A)$ by starting from A . By using such algorithm, in [12, Example 33] it is shown that $\mathcal{V}(\{5\}) = \langle 5, 9, 13, 17, 21 \rangle$. Therefore, by applying Corollary 3.12, we have that $\mathcal{V}_T(\{5\}) = \langle 5, 9, 13, 17, 21 \rangle \cap \langle 5, 7, 9 \rangle = \langle 5, 9, 17, 21 \rangle$.

Example 3.14. An PL-semigroup (see [7]) is a numerical semigroup S fulfilling that $a + b + 1 \in S$ for all $a, b \in S \setminus \{0\}$. Let \mathcal{V} the set formed by all PL-semigroups. In [7] it is shown that \mathcal{V} is a variety and it is given an algorithm to compute $\mathcal{V}(A)$

by starting from A . Let $T = \langle 4, 7, 13 \rangle$ (observe that $T \in \mathcal{V}$). By Theorem 2.9, we know that $\mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\}$ is an R -variety. Let us suppose that we can compute $\mathcal{V}_T(\{4, 7\})$.

From [7, Example 48], we know that $\mathcal{V}(\{4, 7\}) = \langle 4, 7, 9 \rangle$. Thus, by applying Corollary 3.12, we have that $\mathcal{V}_T(\{4, 7\}) = \langle 4, 7, 9 \rangle \cap \langle 4, 7, 13 \rangle = \langle 4, 7, 13 \rangle$.

Let T be a numerical semigroup. We know that, if M is a \mathcal{V}_T -monoid, then there exists a \mathcal{V} -monoid, M' , with minimal \mathcal{V} -system of generators contained in T , such that $M = M' \cap T$. The next result says us that, in this situation, the minimal \mathcal{V} -system of generators of M' is just the minimal \mathcal{V}_T -system of generators of M .

Proposition 3.15. *Let $A \subseteq T$. Then A is the minimal \mathcal{V}_T -system of generators of $\mathcal{V}_T(A)$ if and only if A is the minimal \mathcal{V} -system of generators of $\mathcal{V}(A)$.*

Proof. (Necessity.) Let us suppose that A is not the minimal \mathcal{V} -system of generators of $\mathcal{V}(A)$. That is, there exists $B \subsetneq A$ such that $\mathcal{V}(B) = \mathcal{V}(A)$. Then, from Corollary 3.12, we have that $\mathcal{V}_T(A) = \mathcal{V}(A) \cap T = \mathcal{V}(B) \cap T = \mathcal{V}_T(B)$. Therefore, A is not the minimal \mathcal{V}_T -system of generators of $\mathcal{V}_T(A)$.

(Sufficiency.) Let us suppose that A is not the minimal \mathcal{V}_T -system of generators of $\mathcal{V}_T(A)$. Then $\mathcal{V}_T(B) = \mathcal{V}_T(A)$ for some subset $B \subsetneq A$. On the other hand, due to A is the minimal \mathcal{V} -system of generators of $\mathcal{V}(A)$, from Lemma 3.5, we have an element $a \in A$ such that $a \notin \mathcal{V}(B)$. Consequently, $a \in \mathcal{V}(A) \cap T$ and $a \notin \mathcal{V}(B) \cap T$. Finally, from Corollary 3.12, $\mathcal{V}_T(A) = \mathcal{V}(A) \cap T \neq \mathcal{V}(B) \cap T = \mathcal{V}_T(B)$, which is a contradiction. \square

We finish this section with two examples that illustrate the above proposition.

Example 3.16. Let \mathcal{V} be such as in Example 3.13 and let $T = \langle 4, 6, 7 \rangle$. From [12, Example 26], we know that $\mathcal{V}(\{4, 7, 10\}) = \langle 4, 7, 10, 13 \rangle$ and, moreover, that $\{4\}$ is its minimal \mathcal{V} -system of generators. Then, from Proposition 3.15, $\{4\}$ is the minimal \mathcal{V} -system of generators of $\mathcal{V}_T(\{4, 7, 10\}) = \langle 4, 7, 10, 13 \rangle \cap \langle 4, 6, 7 \rangle$.

Example 3.17. Let \mathcal{V} be such as in Example 3.14 and let $T = \langle 3, 4 \rangle$. From [7, Example 44], we know that $\{3\}$ is the minimal \mathcal{V} -system of generators of $S = \langle 3, 7, 11 \rangle$. Therefore, by Proposition 3.15, $\{3\}$ is the minimal \mathcal{V}_T -system of generators of $S \cap T$.

4 The tree associated to an R -variety

Let V be a non-empty set and let $E \subseteq \{(v, w) \in V \times V \mid v \neq w\}$. It is said that the pair $G = (V, E)$ is a *graph*. In addition, the *vertices* and *edges* of G are the elements of V and E , respectively.

Let $x, y \in V$ and let us suppose that $(v_0, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n)$ is a sequence of different edges such that $v_0 = x$ and $v_n = y$. Then, it is said that such a sequence is a *path (of length n)* connecting x and y .

Let G be a graph. Let us suppose that there exists r , vertex of G , such that it is connected with any other vertex x by a unique path. Then it is said that G is a *tree* and that r is its *root*.

Let x, y be vertices of a tree G and let us suppose that there exists a path that connects x and y . Then it is said that x is a *descendant* of y . Specifically, it is said that x is a *child* of y when (x, y) is an edge of G .

From now on in this section, let \mathcal{R} denote an R -variety. We define the graph $G(\mathcal{R})$ in the following way,

- \mathcal{R} is the set of vertices of $G(\mathcal{R})$;
- $(S, S') \in \mathcal{R} \times \mathcal{R}$ is an edge of $G(\mathcal{R})$ if and only if $S' = S \cup \{F_{\Delta(\mathcal{R})}(S)\}$.

If $S \in \mathcal{R}$, then we can define recurrently (such as we did in Example 3.8) the sequence of elements in \mathcal{R} ,

- $S_0 = S$,
- if $S_i \neq \Delta(\mathcal{R})$, then $S_{i+1} = S_i \cup \{F_{\Delta(\mathcal{R})}(S_i)\}$.

Thus, we obtain a chain (of elements in \mathcal{R}) $S = S_0 \subsetneq S_1 \subsetneq \dots \subsetneq S_n = \Delta(\mathcal{R})$ such that (S_i, S_{i+1}) is an edge of $G(\mathcal{R})$ for all $i \in \{0, \dots, n-1\}$. We will denote by $C_{\mathcal{R}}(S)$ the set $\{S_0, S_1, \dots, S_n\}$ and will say that it is the *chain of S in \mathcal{R}* . The next result is easy to prove.

Proposition 4.1. *$G(\mathcal{R})$ is a tree with root $\Delta(\mathcal{R})$.*

Observe that, in order to recurrently construct $G(\mathcal{R})$ starting from $\Delta(\mathcal{R})$, it is sufficient to compute the children of each vertex of $G(\mathcal{R})$. Let us also observe that, if T is a child of S , then $S = T \cup \{F_{\Delta(\mathcal{R})}(T)\}$. Therefore, $T = S \setminus \{F_{\Delta(\mathcal{R})}(T)\}$. Thus, if T is a child of S , then there exists an integer $x > F_{\Delta(\mathcal{R})}(S)$ such that $T = S \setminus \{x\}$. As a consequence of Propositions 3.9 and 4.1, and defining $F_{\Delta(\mathcal{R})}(\Delta(\mathcal{R})) = -1$, we have the following result.

Theorem 4.2. *The graph $G(\mathcal{R})$ is a tree with root equal to $\Delta(\mathcal{R})$. Moreover, the set formed by the children of a vertex $S \in \mathcal{R}$ is $\{S \setminus \{x\} \mid x \text{ is an element of the minimal } \mathcal{R}\text{-system of generators of } S \text{ and } x > F_{\Delta(\mathcal{R})}(S)\}$.*

We can reformulate the above theorem in the following way.

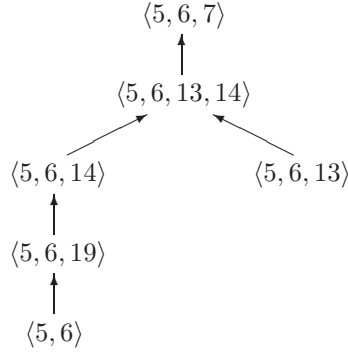
Corollary 4.3. *The graph $G(\mathcal{R})$ is a tree with root equal to $\Delta(\mathcal{R})$. Moreover, the set formed by the children of a vertex $S \in \mathcal{R}$ is $\{S \setminus \{x\} \mid x \in \text{msg}(S), x > F_{\Delta(\mathcal{R})}(S) \text{ and } S \setminus \{x\} \in \mathcal{R}\}$.*

We illustrate the previous results with an example.

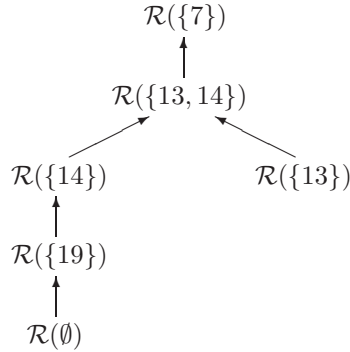
Example 4.4. We are going to build the R -variety $\mathcal{R} = [\langle 5, 6 \rangle, \langle 5, 6, 7 \rangle] = \{S \mid S \text{ is a numerical semigroup and } \langle 5, 6 \rangle \subseteq S \subseteq \langle 5, 6, 7 \rangle\}$. Observe that, if $S \in \mathcal{R}$ and $x \in \text{msg}(S)$, then $S \setminus \{x\} \in \mathcal{R}$ if and only if $x \notin \{5, 6\}$. Moreover, the maximum of \mathcal{R} is $\Delta = \langle 5, 6, 7 \rangle$. By applying Corollary 4.3, we can recurrently build $G(\mathcal{R})$ in the following way.

- $\langle 5, 6, 7 \rangle$ has got a unique child, which is $\langle 5, 6, 7 \rangle \setminus \{7\} = \langle 5, 6, 13, 14 \rangle$. Moreover, $F_{\Delta}(\langle 5, 6, 13, 14 \rangle) = 7$.
- $\langle 5, 6, 13, 14 \rangle$ has got two children, which are $\langle 5, 6, 13, 14 \rangle \setminus \{13\} = \langle 5, 6, 14 \rangle$ and $\langle 5, 6, 13, 14 \rangle \setminus \{14\} = \langle 5, 6, 13 \rangle$. Moreover, $F_{\Delta}(\langle 5, 6, 14 \rangle) = 13$ and $F_{\Delta}(\langle 5, 6, 13 \rangle) = 14$.
- $\langle 5, 6, 13 \rangle$ has not got children.
- $\langle 5, 6, 14 \rangle$ has got a unique child, which is $\langle 5, 6, 14 \rangle \setminus \{14\} = \langle 5, 6, 19 \rangle$. Moreover, $F_{\Delta}(\langle 5, 6, 19 \rangle) = 14$.
- $\langle 5, 6, 19 \rangle$ has got a unique child, which is $\langle 5, 6, 19 \rangle \setminus \{19\} = \langle 5, 6 \rangle$. Moreover, $F_{\Delta}(\langle 5, 6 \rangle) = 19$.
- $\langle 5, 6 \rangle$ has not got children.

Therefore, in this situation, $G(\mathcal{R})$ is given by the next diagram.



Observe that, if we represent the vertices of $G(\mathcal{R})$ using their minimal \mathcal{R} -systems of generators, then we have that $G(\mathcal{R})$ is given by the following diagram.



Let us observe that the R -variety $\mathcal{R} = [\langle 5, 6 \rangle, \langle 5, 6, 7 \rangle]$ depict in the above example is finite and, therefore, we have been able to build all its elements in a finite number of steps. If the R -variety is infinite, then it is not possible such situation. However, as a consequence of Theorem 4.2, we can show an algorithm in order to compute all the elements of the R -variety when the genus is fixed.

Algorithm 4.5. INPUT: A positive integer g .

OUTPUT: $\{S \in \mathcal{R} \mid g(S) = g\}$.

- (1) If $g < g(\Delta(\mathcal{R}))$, then return \emptyset .
- (2) Set $A = \{\Delta(\mathcal{R})\}$ and $i = g(\Delta(\mathcal{R}))$.
- (3) If $i = g$, then return A .
- (4) For each $S \in A$, compute the set B_S formed by all elements of the minimal \mathcal{R} -system of generators of S that are greater than $F_{\Delta(\mathcal{R})}(S)$.
- (5) If $\bigcup_{S \in A} B_S = \emptyset$, then return \emptyset .
- (6) Set $A = \bigcup_{S \in A} \{S \setminus \{x\} \mid x \in B_S\}$, $i = i + 1$, and go to (3).

We illustrate the operation of this algorithm with an example.

Example 4.6. Let $\Delta = \langle 4, 6, 7 \rangle = \{0, 4, 6, 7, 8, 10, \rightarrow\}$. It is clear that $g(\Delta) = 5$. Let $\mathcal{R} = \{S \mid S \text{ is a numerical semigroup and } \{4, 6\} \subseteq S \subseteq \Delta\}$. We have that \mathcal{R} is an infinite R -variety because $\langle 4, 6, 2k+1 \rangle \in \mathcal{R}$ for all $k \in \{5, \rightarrow\}$. By using Algorithm 4.5, we are going to compute the set $\{S \in \mathcal{R} \mid g(S) = 8\}$.

- $A = \{\Delta\}$, $i = 5$.
- $B_\Delta = \{7\}$.
- $A = \{\langle 4, 6, 11, 13 \rangle\}$, $i = 6$.
- $B_{\langle 4, 6, 11, 13 \rangle} = \{11, 13\}$.
- $A = \{\langle 4, 6, 13, 15 \rangle, \langle 4, 6, 11 \rangle\}$, $i = 7$.
- $B_{\langle 4, 6, 13, 15 \rangle} = \langle 13, 15 \rangle$ and $B_{\langle 4, 6, 11 \rangle} = \emptyset$.
- $A = \{\langle 4, 6, 15, 17 \rangle, \langle 4, 6, 13 \rangle\}$, $i = 8$.
- The algorithm returns $\{\langle 4, 6, 15, 17 \rangle, \langle 4, 6, 13 \rangle\}$.

Our next purpose in this section will be to show that, if \mathcal{R} is an R -variety and $T \in \mathcal{R}$, then the set formed by all the descendants of T in the tree $G(\mathcal{R})$ is also an R -variety. It is clear that, if $S, T \in \mathcal{R}$, then S is a descendant of T if and only if $T \in C_{\mathcal{R}}(S)$. Therefore, we can establish the following result.

Lemma 4.7. *Let \mathcal{R} be an R -variety and $S, T \in \mathcal{R}$. Then S is a descendant of T if and only if there exists $n \in \mathbb{N}$ such that $T = S \cup \{x \in \Delta(\mathcal{R}) \mid x \geq n\}$.*

As an immediate consequence of the above lemma, we have the next one.

Lemma 4.8. *Let \mathcal{R} be an R -variety and $S, T \in \mathcal{R}$ such that $S \neq T$. If S is a descendant of T , then $F_{\Delta(\mathcal{R})}(S) = F_T(S)$.*

Now we are ready to show the announced result.

Theorem 4.9. *Let \mathcal{R} be an R -variety and $T \in \mathcal{R}$. Then*

$$\mathcal{D}(T) = \{S \in \mathcal{R} \mid S \text{ is a descendant of } T \text{ in the tree } G(\mathcal{R})\}$$

is an R -variety.

Proof. Clearly, T is the maximum of $\mathcal{D}(T)$. Let us see that, if $S_1, S_2 \in \mathcal{D}(T)$, then $S_1 \cap S_2 \in \mathcal{D}(T)$. Since, from Lemma 4.7, we know that there exist $n_1, n_2 \in \mathbb{N}$ such that $T = S_i \cup \{x \in \Delta(\mathcal{R}) \mid x \geq n_i\}$, $i = 1, 2$, it is sufficient to show that $T = (S_1 \cap S_2) \cup \{x \in \Delta(\mathcal{R}) \mid x \geq \min\{n_1, n_2\}\}$. It is obvious that $(S_1 \cap S_2) \cup \{x \in \Delta(\mathcal{R}) \mid x \geq \min\{n_1, n_2\}\} \subseteq T$. Let us see now the opposite inclusion. For that, let $t \in T$ such that $t \notin S_1 \cap S_2$. Then $t \notin S_1$ or $t \notin S_2$ and, therefore, $t \in \{x \in \Delta(\mathcal{R}) \mid x \geq n_1\}$ or $t \in \{x \in \Delta(\mathcal{R}) \mid x \geq n_2\}$. Thereby, $t \in \{x \in \Delta(\mathcal{R}) \mid x \geq \min\{n_1, n_2\}\}$. Consequently, $T \subseteq (S_1 \cap S_2) \cup \{x \in \Delta(\mathcal{R}) \mid x \geq \min\{n_1, n_2\}\}$. By applying again Lemma 4.7, we can assert that $S_1 \cap S_2 \in \mathcal{D}(T)$.

Finally, let $S \in \mathcal{D}(T)$ such that $S \neq T$. From Lemma 4.8, $S \cup \{F_T(S)\} = S \cup \{F_{\Delta(\mathcal{R})}(S)\} \in \mathcal{R}$ and, in consequence, $S \cup \{F_T(S)\} \in \mathcal{D}(T)$. \square

From the previous comment to [8, Example 7], we know that, if \mathcal{V} is a variety and $T \in \mathcal{V}$, then $\mathcal{D}(T)$ is a pseudo-variety and, moreover, every pseudo-variety can be obtained in this way. Therefore, there exist R -varieties which are not the set formed by all the descendants of an element belonging to a variety.

The following result shows that an R -variety can be obtained as the set formed by intersecting all the descendants, of an element belonging to a variety, with a numerical semigroup.

Corollary 4.10. *Let \mathcal{V} be a variety, let $\Delta \in \mathcal{V}$, and let T be a numerical semigroup. Let $\mathcal{D}(\Delta) = \{S \mid S \text{ is a descendant of } \Delta \text{ in } G(\mathcal{V})\}$ and let $\mathcal{D}(\Delta, T) = \{S \cap T \mid S \in \mathcal{D}(\Delta)\}$. Then $\mathcal{D}(\Delta, T)$ is an R -variety. Moreover, every R -variety can be obtained in this way.*

Proof. If \mathcal{V} is a variety, then \mathcal{V} is an R -variety and, by applying Theorem 4.9, we have that $\mathcal{D}(\Delta)$ is an R -variety. From Corollary 2.10, we conclude that $\mathcal{D}(\Delta, T)$ is an R -variety.

If \mathcal{R} is an R -variety, by Theorem 2.9, we know that there exist a variety \mathcal{V} and a numerical semigroup T such that $\mathcal{R} = \{S \cap T \mid S \in \mathcal{V}\}$. Now, it is clear that $\mathcal{V} = \mathcal{D}(\mathbb{N}) = \{S \mid S \text{ is a descendant of } \mathbb{N} \text{ in } G(\mathcal{V})\}$. Therefore, we have that $\mathcal{R} = \{S \cap T \mid S \in \mathcal{D}(\mathbb{N})\} = \mathcal{D}(\mathbb{N}, T)$. \square

In the next result we see that the above corollary is also true when we write pseudo-variety instead of variety.

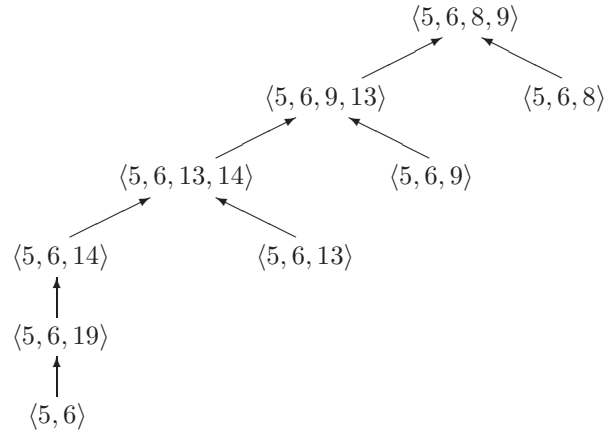
Corollary 4.11. *Let \mathcal{P} be a pseudo-variety, let $\Delta \in \mathcal{P}$, and let T be a numerical semigroup. Let $\mathcal{D}(\Delta) = \{S \mid S \text{ is a descendant of } \Delta \text{ in } G(\mathcal{P})\}$ and let $\mathcal{D}(\Delta, T) = \{S \cap T \mid S \in \mathcal{D}(\Delta)\}$. Then $\mathcal{D}(\Delta, T)$ is an R -variety. Moreover, every R -variety can be obtained in this way.*

Proof. If \mathcal{P} is a pseudo-variety, then \mathcal{P} is an R -variety and, by applying Theorem 4.9, we have that $\mathcal{D}(\Delta)$ is an R -variety as well. Now, from Corollary 2.10, we have that $\mathcal{D}(\Delta, T)$ is an R -variety.

That every R -variety can be obtained in this way is an immediate consequence of Corollary 4.10 and having in mind that each variety is a pseudo-variety. \square

We conclude this section by illustrating the above corollary with an example.

Example 4.12. Let \mathcal{P} the pseudo-variety which appear in Example 2.12. In [8, Example 7] it is shown that $G(\mathcal{P})$ is given by the next subtree.



By applying Corollary 4.11, we have that, if T is a numerical semigroup, then $\mathcal{R} = \{S \cap T \mid S \in \mathcal{D}(\langle 5, 6, 13, 14 \rangle)\}$ is an R -variety. Let us observe that $\mathcal{D}(\langle 5, 6, 13, 14 \rangle) = \{\langle 5, 6, 13, 14 \rangle, \langle 5, 6, 14 \rangle, \langle 5, 6, 13 \rangle, \langle 5, 6, 19 \rangle, \langle 5, 6 \rangle\}$.

5 The smallest R -variety containing a family of numerical semigroups

In [11, Proposition 2] it is proved that the intersection of varieties is a variety. As a consequence of this, we have that there exists the smallest variety which contains a given family of numerical semigroups.

On the other hand, in [8] was shown that, in general, the intersection of pseudo-varieties is not a pseudo-variety. Nevertheless, in [8, Theorem 4] it is proved that there exists the smallest pseudo-variety which contains a given family of numerical semigroups.

Our first objective in this section will be to show that, in general, we cannot talk about the smallest R -variety which contains a given family of numerical semigroups.

Lemma 5.1. *Let \mathcal{F} be a family of numerical semigroups and let Δ be a numerical semigroup such that $S \subseteq \Delta$ for all $S \in \mathcal{F}$. Then there exists an R -variety \mathcal{R} such that $\mathcal{F} \subseteq \mathcal{R}$ and $\max(\mathcal{R}) = \Delta$.*

Proof. Let $\mathcal{R} = \{S \mid S \text{ is a numerical semigroup and } S \subseteq \Delta\}$. From Item 1 in Example 2.3, we have that \mathcal{R} is an R -variety. Now, it is trivial that $\mathcal{F} \subseteq \mathcal{R}$ and $\max(\mathcal{R}) = \Delta$. \square

The proof of the next lemma is straightforward and we can omit it.

Lemma 5.2. *Let $\{\mathcal{R}_i\}_{i \in I}$ be a family of R -varieties such that $\max(\mathcal{R}_i) = \Delta$ for all $i \in I$. Then $\bigcap_{i \in I} \mathcal{R}_i$ is an R -variety and $\max(\bigcap_{i \in I} \mathcal{R}_i) = \Delta$.*

The following result says us that there exists the smallest R -variety which contains a given family of numerical semigroups and has a certain maximum.

Proposition 5.3. *Let \mathcal{F} be a family of numerical semigroups and let Δ be a numerical semigroup such that $S \subseteq \Delta$ for all $S \in \mathcal{F}$. Then there exists the smallest R -variety which contains \mathcal{F} and with maximum equal to Δ .*

Proof. Let \mathcal{R} be the intersection of all the R -varieties containing \mathcal{F} and with maximum equal to Δ . From Lemmas 5.1 and 5.2 we have the conclusion. \square

We will denote by $\mathcal{R}(\mathcal{F}, \Delta)$ the R -variety given by Proposition 5.3. Now we are interested in describe the elements of such an R -variety.

Lemma 5.4. *Let $S_1, S_2, \dots, S_n, \Delta$ be numerical semigroups such that $S_i \subseteq \Delta$ for all $i \in \{1, \dots, n\}$. Then $F_\Delta(S_1 \cap \dots \cap S_n) = \max\{F_\Delta(S_1), \dots, F_\Delta(S_n)\}$.*

Proof. We have that $F_\Delta(S_1 \cap \dots \cap S_n) = \max(\Delta \setminus (S_1 \cap \dots \cap S_n)) = \max((\Delta \setminus S_1) \cup \dots \cup (\Delta \setminus S_n)) = \max\{\max(\Delta \setminus S_1), \dots, \max(\Delta \setminus S_n)\} = \max\{F_\Delta(S_1), \dots, F_\Delta(S_n)\}$. \square

Let us recall that, if S and Δ are numerical semigroups such that $S \subseteq \Delta$, then we defined $C(S, \Delta)$ in Example 3.8 (that is, the chain of S restricted to Δ). If \mathcal{F} is a family of numerical semigroups such that $S \subseteq \Delta$ for all $S \in \mathcal{F}$, then we will denote by $C(\mathcal{F}, \Delta)$ the set $\bigcup_{S \in \mathcal{F}} C(S, \Delta)$.

Theorem 5.5. *Let \mathcal{F} be a family of numerical semigroups and let Δ be a numerical semigroup such that $S \subseteq \Delta$ for all $S \in \mathcal{F}$. Then $\mathcal{R}(\mathcal{F}, \Delta)$ is the set formed by all the finite intersections of elements in $C(\mathcal{F}, \Delta)$.*

Proof. Let $\mathcal{R} = \{S_1 \cap \dots \cap S_n \mid n \in \mathbb{N} \setminus \{0\} \text{ and } S_1, \dots, S_n \in C(\mathcal{F}, \Delta)\}$. Having in mind that $\mathcal{R}(\mathcal{F}, \Delta)$ is an R -variety which contains \mathcal{F} and with maximum equal to Δ , we easily deduce that $\mathcal{R} \subseteq \mathcal{R}(\mathcal{F}, \Delta)$.

Let us see now that \mathcal{R} is an R -variety. On the one hand, it is clear that $\Delta = \max(\mathcal{R})$ and that, if $S, T \in \mathcal{R}$, then $S \cap T \in \mathcal{R}$. On the other hand, let $S \in \mathcal{R}$ such that $S \neq \Delta$. Then $S = S_1 \cap \dots \cap S_n$ for some $S_1, \dots, S_n \in C(\mathcal{F}, \Delta)$. Now, from Lemma 5.4, we have that $F_\Delta(S) = \max\{F_\Delta(S_1), \dots, F_\Delta(S_n)\}$ and, thus, $F_\Delta(S_i) \leq F_\Delta(S)$ for all $i \in \{1, \dots, n\}$. Let us observe that, if $F_\Delta(S) > F_\Delta(S_i)$, then $S_i \cup \{F_\Delta(S)\} = S_i$. Moreover, if $F_\Delta(S) = F_\Delta(S_i)$, then we get $S_i \cup \{F_\Delta(S)\} = S_i \cup \{F_\Delta(S_i)\} \in C(\mathcal{F}, \Delta)$. Therefore, $S_i \cup \{F_\Delta(S)\} \in C(\mathcal{F}, \Delta)$ for all $i \in \{1, \dots, n\}$. Since $S \cup \{F_\Delta(S)\} = (S_1 \cup \{F_\Delta(S)\}) \cap \dots \cap (S_n \cup \{F_\Delta(S)\})$, then $S \cup \{F_\Delta(S)\} \in \mathcal{R}$. Consequently, \mathcal{R} is an R -variety.

Finally, since \mathcal{R} is an R -variety which contains \mathcal{F} and with maximum equal to Δ , then $\mathcal{R}(\mathcal{F}, \Delta) \subseteq \mathcal{R}$ and, thereby, we conclude that $\mathcal{R} = \mathcal{R}(\mathcal{F}, \Delta)$. \square

Let us observe that, if \mathcal{F} is a finite family, then $C(\mathcal{F}, \Delta)$ is a finite set and, therefore, $\mathcal{R}(\mathcal{F}, \Delta)$ is a finite R -variety.

Lemma 5.6. *Let \mathcal{R} and \mathcal{R}' be two R -varieties. If $\mathcal{R} \subseteq \mathcal{R}'$, then $\Delta(\mathcal{R}) \subseteq \Delta(\mathcal{R}')$.*

Proof. If $\mathcal{R} \subseteq \mathcal{R}'$, then $\Delta(\mathcal{R}) \in \mathcal{R}'$. Therefore, $\Delta(\mathcal{R}) \subseteq \Delta(\mathcal{R}')$. \square

The next example shows us that, in general, we cannot talk about the smallest R -variety which contains a given family of numerical semigroups.

Example 5.7. Let $\mathcal{F} = \{\langle 5, 6 \rangle, \langle 5, 7 \rangle\}$. As a consequence of Lemma 5.6, the candidate to be the smallest R -variety which contains \mathcal{F} must have as maximum the numerical semigroup $\langle 5, 6, 7 \rangle$ (that is, the smallest numerical semigroup containing $\langle 5, 6 \rangle$ and $\langle 5, 7 \rangle$). Thus, the candidate to be the smallest R -variety which contains \mathcal{F} is $\mathcal{R}(\mathcal{F}, \langle 5, 6, 7 \rangle)$.

Let us see now that $\mathcal{R}(\mathcal{F}, \langle 5, 6, 7 \rangle) \not\subseteq \mathcal{R}(\mathcal{F}, \langle 5, 6, 7, 8 \rangle)$ and, in this way, that there does not exist the smallest R -variety which contains \mathcal{F} . In order to do it, we will show that $\langle 5, 6, 7 \rangle \notin \mathcal{R}(\mathcal{F}, \langle 5, 6, 7, 8 \rangle)$. In fact, by applying Theorem 5.5, if $\langle 5, 6, 7 \rangle \in \mathcal{R}(\mathcal{F}, \langle 5, 6, 7, 8 \rangle)$, then we deduce that there exist $S_1 \in C(\langle 5, 6 \rangle, \langle 5, 6, 7, 8 \rangle)$ and $S_2 \in C(\langle 5, 7 \rangle, \langle 5, 6, 7, 8 \rangle)$ such that $S_1 \cap S_2 = \langle 5, 6, 7 \rangle$. Since $S_1 \in C(\langle 5, 6 \rangle, \langle 5, 6, 7, 8 \rangle)$, then there exists $n_1 \in \mathbb{N}$ such that $S_1 = \langle 5, 6 \rangle \cup \{x \in \langle 5, 6, 7, 8 \rangle \mid x \geq n_1\}$. Moreover, $\langle 5, 6, 7 \rangle \subseteq S_1$ and, thereby, $n_1 \leq 7$. Consequently, $8 \in S_1$. By an analogous reasoning, we have that $8 \in S_2$ too. Consequently, $8 \in S_1 \cap S_2 = \langle 5, 6, 7 \rangle$, which is false.

Let \mathcal{R} be an R -variety. We will say that \mathcal{F} (subset of \mathcal{R}) is a *system of generators* of \mathcal{R} if $\mathcal{R} = \mathcal{R}(\mathcal{F}, \Delta(\mathcal{R}))$. Let us observe that, in such a case, \mathcal{R} is the smallest R -variety which contains \mathcal{F} and with maximum equal to $\Delta(\mathcal{R})$.

We will say that an R -variety, \mathcal{R} , is finitely generated if there exists a finite set $\mathcal{F} \subseteq \mathcal{R}$ such that $\mathcal{R} = \mathcal{R}(\mathcal{F}, \Delta(\mathcal{R}))$ (that is, if \mathcal{R} has a finite system of generators). As a consequence of Theorem 5.5, we have the following result.

Corollary 5.8. *An R -variety is finitely generated if and only if it is finite.*

From now on, \mathcal{F} will denote a family of numerical semigroups and Δ will denote a numerical semigroup such that $S \subseteq \Delta$ for all $S \in \mathcal{F}$. Our purpose is to give a method in order to compute the minimal $\mathcal{R}(\mathcal{F}, \Delta)$ -system of generators of a $\mathcal{R}(\mathcal{F}, \Delta)$ -monoid by starting from \mathcal{F} and Δ .

If $A \subseteq \Delta$, then for each $S \in \mathcal{F}$ we define

$$\alpha(S) = \begin{cases} S, & \text{if } A \subseteq S, \\ S \cup \{x \in \Delta \mid x \geq x_S\}, & \text{if } A \not\subseteq S, \end{cases}$$

where $x_S = \min\{a \in A \mid a \notin S\}$.

As a consequence of Lemma 3.2 and Theorem 5.5, we have the next result.

Lemma 5.9. *The $\mathcal{R}(\mathcal{F}, \Delta)$ -monoid generated by A is $\bigcap_{S \in \mathcal{F}} \alpha(S)$.*

Recalling that $\mathcal{R}(\mathcal{F}, \Delta)(A)$ denotes the $\mathcal{R}(\mathcal{F}, \Delta)$ -monoid generated by A , we have the following result.

Proposition 5.10. *If $A \subseteq \Delta$, then $B = \{x_S \mid S \in \mathcal{F} \text{ and } A \not\subseteq S\}$ is the minimal $\mathcal{R}(\mathcal{F}, \Delta)$ -system of generators of $\mathcal{R}(\mathcal{F}, \Delta)(A)$.*

Proof. Let us observe that, if $S \in \mathcal{F}$, then $A \subseteq S$ if and only if $B \subseteq S$. Moreover, if $A \not\subseteq S$, then $\min\{a \in A \mid a \notin S\} = \min\{b \in B \mid b \notin S\}$. Therefore, by applying Lemma 5.9, we have that $\mathcal{R}(\mathcal{F}, \Delta)(A) = \mathcal{R}(\mathcal{F}, \Delta)(B)$. Consequently, in order to prove that B is the minimal $\mathcal{R}(\mathcal{F}, \Delta)$ -system of generators of $\mathcal{R}(\mathcal{F}, \Delta)(A)$, will be enough to see that, if $C \subsetneq B$, then $\mathcal{R}(\mathcal{F}, \Delta)(C) \neq \mathcal{R}(\mathcal{F}, \Delta)(A)$.

In effect, if $C \subsetneq B$, then there exists $S \in \mathcal{F}$ such that $x_S \notin C$ and, thereby, we have that $C \subseteq S$ or that $\min\{c \in C \mid c \notin S\} > x_S$. Now, by applying once more time Lemma 5.9, we easily deduce that $x_S \notin \mathcal{R}(\mathcal{F}, \Delta)(C)$. Since $x_S \in B \subseteq A$, then we get that $A \not\subseteq \mathcal{R}(\mathcal{F}, \Delta)(C)$ and, therefore, $\mathcal{R}(\mathcal{F}, \Delta)(C) \neq \mathcal{R}(\mathcal{F}, \Delta)(A)$. \square

As an immediate consequence of the above proposition we have the next result.

Corollary 5.11. *Every $\mathcal{R}(\mathcal{F}, \Delta)$ -monoid has $\mathcal{R}(\mathcal{F}, \Delta)$ -range less than or equal to the cardinality of \mathcal{F} .*

We will finish this section by illustrating its content with an example.

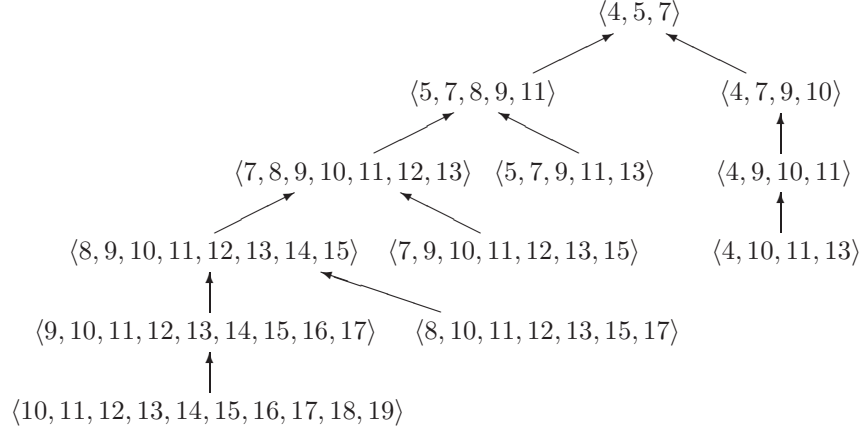
Example 5.12. Let $\mathcal{F} = \{\langle 5, 7, 9, 11, 13 \rangle, \langle 4, 10, 11, 13 \rangle\}$ and $\Delta = \langle 4, 5, 7 \rangle$. We are going to compute the tree $G(\mathcal{R}(\mathcal{F}, \Delta))$.

First of all, to compute the minimal $\mathcal{R}(\mathcal{F}, \Delta)$ -system of generators of $\langle 4, 5, 7 \rangle$, we apply Proposition 5.10 with $A = \{4, 5, 7\}$. Since $x_{\langle 5, 7, 9, 11, 13 \rangle} = 4$ and $x_{\langle 4, 10, 11, 13 \rangle} = 5$, then $\{4, 5\}$ is such a minimal $\mathcal{R}(\mathcal{F}, \Delta)$ -system. Now, because $F_\Delta(\langle 4, 5, 7 \rangle) = -1$, and by applying Theorem 4.2, we get that $\langle 4, 5, 7 \rangle$ has two children, $\langle 4, 5, 7 \rangle \setminus \{4\} = \langle 5, 7, 8, 9, 11 \rangle$ (with $F_\Delta(\langle 5, 7, 8, 9, 11 \rangle) = 4$) and $\langle 4, 5, 7 \rangle \setminus \{5\} = \langle 4, 7, 9, 10 \rangle$ (with $F_\Delta(\langle 4, 7, 9, 10 \rangle) = 5$).

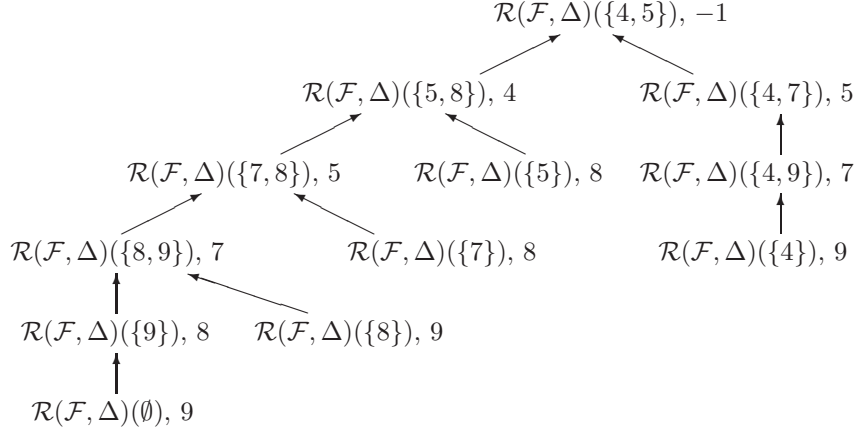
Now, if we take $A = \{5, 7, 8, 9, 11\}$ in Proposition 5.10, then we have that $x_{\langle 5, 7, 9, 11, 13 \rangle} = 8$ and $x_{\langle 4, 10, 11, 13 \rangle} = 5$. Thus, we conclude that $\{5, 8\}$ is the minimal $\mathcal{R}(\mathcal{F}, \Delta)$ -system of $\langle 5, 7, 8, 9, 11 \rangle$. Moreover, since $F_\Delta(\langle 5, 7, 8, 9, 11 \rangle) = 4$, then Theorem 4.2 asserts that $\langle 5, 7, 8, 9, 11 \rangle \setminus \{5\} = \langle 7, 8, 9, 10, 11, 12, 13 \rangle$ (with $F_\Delta(\langle 7, 8, 9, 10, 11, 12, 13 \rangle) = 5$) and $\langle 5, 7, 8, 9, 11 \rangle \setminus \{8\} = \langle 5, 7, 9, 11, 13 \rangle$ (with $F_\Delta(\langle 5, 7, 9, 11, 13 \rangle) = 8$) are the two children of $\langle 5, 7, 8, 9, 11 \rangle$.

With $A = \{4, 7, 9, 10\}$, we get that $\{4, 7\}$ is the minimal $\mathcal{R}(\mathcal{F}, \Delta)$ -system of $\langle 4, 7, 9, 10 \rangle$. By recalling that $F_\Delta(\langle 4, 7, 9, 10 \rangle) = 5$, we conclude that $\langle 4, 7, 9, 10 \rangle$ has only one child, that is $\langle 4, 7, 9, 10 \rangle \setminus \{7\} = \langle 4, 9, 10, 11 \rangle$ (with $F_\Delta(\langle 4, 9, 10, 11 \rangle) = 7$).

By repeating the above process, we get the whole tree $G(\mathcal{R}(\mathcal{F}, \Delta))$.



Now, we are going to represent the vertices of $G(\mathcal{R}(\mathcal{F}, \Delta))$ using their minimal $\mathcal{R}(\mathcal{F}, \Delta)$ -systems of generators. Moreover, we add to each vertex the corresponding Frobenius number restricted to Δ . Thus, we clarify all the steps to build the tree $G(\mathcal{R}(\mathcal{F}, \Delta))$.



6 Conclusion

We have been able to give a structure to certain families of numerical semigroups which are not (Frobenius) varieties or (Frobenius) pseudo-varieties. For that we have generalized the concept of Frobenius number to the concept of restricted Frobenius number and, then, we have defined the R -varieties (or (Frobenius) restricted variety).

After studying relations among varieties, pseudo-varieties, and R -varieties, we have introduced the concepts of R -monoid and minimal R -system of generators of a R -monoid, which lead to associate a tree with each R -variety and, in consequence, obtain recurrently all the elements of an R -variety.

Finally, although in general it is not possible to define the smallest R -variety that contains a given family \mathcal{F} of numerical semigroups, we have been able to

give an alternative when we fix in advance the maximum of the smallest R -variety.

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